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A nonconvex dissipative system and its applications (I)

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Abstract In order to study the uniformly translating solution of some non-linear evolution equations such as the complex Ginzburg–Landau equation, this paper presents a qualitative analysis to a Duffing–van der Pol non-linear oscillator. Monotonic property of the bounded exact solution is established based on the construction of a convex domain. Under certain parametric choices, one first integral to the Duffing–van der Pol non-linear system is obtained by using the Lie symmetry analysis, which constitutes one of the bases for further work of obtaining uniformly translating solutions of the complex Ginzburg–Landau equation.

Keywords Ginzburg–Landau equation · Autonomous system · First integral · Oscillator · Equilibrium point · Lie symmetry.

AMS (MOS) Subject Classification 34C05 · 34C14 · 34C20 · 35B40

1 Introduction

The word 'zawgyne' is consistently translated into the English language as 'pairs'. In scientific literature its meaning is extended to incorporate such concepts as 'duality', 'complementarity', and several other concepts reflecting conjugate and/or reciprocal

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Dedicated to Professor G. Strang on the occasion of his 70th birthday

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properties. Today within many respective disciplines, from the natural, engineering, biological and social sciences, complementarity and duality are becoming multi-disciplinary topics that play a crucial role in many physical, chemical and biological phenomena, and form the basis for solving many underlying non-convex or global optimization problems that arise in sciences and engineering [1,2]. In physics, complementarity is a basic principle of quantum theory, and refers to effects such as the wave–particle duality, in which different measurements made on a system reveal it to have either particle-like or wave-like properties. Duality exists everywhere and in nature is amazingly beautiful. In our daily life, it means the sort of harmony of two opposite or complementary parts by which they integrate into a whole. The theory of duality is a vast subject, particularly in natural sciences. Mathematics lies at its root. It has beautiful theoretical properties, realistic applications, and pleasing relationships with the existing fundamental theories [3–9].

In many phenomena described by non-linear systems which arise in the areas of physics, engineering and convex/non-convex or global optimization, the direct/indirect interaction is exhibited between the general complementarity, duality principles and the special structures of particular problems to which they can be applied (recent developments in applied mathematical modeling, analysis and optimization allow us analyze models without restrictive assumptions, like differentiability of convexity). This interaction can lead to effective computational methods that take advantage of the structure presented in the problem, with particular interests in the numerical methods and mathematical computations [10–12]. The case of non-linear dissipative systems and their applications in modern nonlinear mechanics is a typical example. All the equations which are known to be exactly integrable are of the Hamiltonian or purely dispersive type. That is, dissipation is discarded entirely or at best taken into account perturbatively in the limit of very weak dissipation. On the other hand, for many macroscopic phenomena in physics dissipation is not just a small perturbation, but plays a significant role in the determination of the dynamical behavior. Therefore, it seems natural to ask whether it is possible to have a strongly dissipative system whose solutions share at least some of the properties which those prototype systems known to be exactly integrable; whether it is, for example, possible to have particle-like solutions of an equation of evolution, which collide and interpenetrate, but which are unchanged in speed, size, and shape after collision.

In this paper, we consider a Duffing–van der Pol oscillator with two polynomial non-linearities

$$\ddot{x} + (\alpha + \beta x^2)\dot{x} - \gamma x + x^3 - \mu x^5 = 0, \tag{1}$$

where α, β, γ and μ are real constants, and an overdot denotes differentiation with respect to time. Equation 1 is the well-studied but still challenging problem in non-convex dynamics with autonomous version which arises in many contexts of physics and engineering. For example, several special cases of Eq. 1 are widely applied to the study of oscillations of a rigid pendulum undergoing with moderately large amplitude motion [13], vibrations of a buckled beam and so on [14–16]. It has provided a useful paradigm for studies in non-linear oscillations and chaotic dynamical systems with a forced term on the right-hand-side, dating back to the development of approximate analytical methods based on perturbative ideas [13], and still continues, with the advent of fast numerical integration by computer, to be used as an archetypal illustration of chaos [14,16–19]. In particular, we see that

(i) when $\mu = 0$ in (1), then Eq. 1 becomes

$$\ddot{x} + (\alpha + \beta x^2)\dot{x} - \gamma x + x^3 = 0, \tag{2}$$

which is the standard form of the Duffing–van der Pol oscillator. Equation 2 arises in a model describing propagation of voltage pulses along a neuronal axon and certain flow induced structural vibration problems in which the structural nonlinearities act to maintain overall stability. It has received a lot of attention by many authors. A vast amount of literature exists on this equation, for details see, for example, [13,14,20,21] and references therein.

(ii) when $\beta = 0$ and $\mu = 0$, then

$$\ddot{x} + \alpha\dot{x} - \gamma x + x^3 = 0, \tag{3}$$

and (3) incorporates the force-free Duffing equation whose integrability, non-integrability properties and existence of the limit cycle are well known [13,22], where α is the coefficient of viscous damping and the term $-\mu x + x^3$ represents the nonlinear restoring force, acting like a hard spring.

(iii) when the two higher-order non-linearities are absent in (1), it becomes the well-known van der Pol equation [23,24]

$$\ddot{x} + (\alpha + \beta x^2)\dot{x} - \gamma x = 0, \tag{4}$$

which was suggested by van der Pol as a non-linear model for the study of the electrical circuit. The sign of the damping term depends on the magnitude of the displacement x.

As far as the integrability properties of Eq. 1 are concerned not much progress has been made mainly due to the fact that even one of its particular case (2) does not pass the Painlevé test as it admits a movable algebraic branch point and a local Laurent expansion in the form [25]

$$x(t) = \frac{3}{2\beta} \cdot \tau^{-1/2} + \frac{3}{2\beta} \cdot \left(\frac{3}{2\beta} - \frac{\alpha}{2}\right)\tau^{1/2} + a_3\tau + \cdots,$$

where $\tau = t - t_0$ and t_0 and a_3 are arbitrary constants. Therefore, it is natural to ask under what particular conditions, Eq. 1 is integrable and its exact solution can be expressed in the functional form.

Motivated by potential applications in physics, engineering, biology and communication theory, Eqs. 1–4 have received an increasing interest, and hence the literature on these equations is very comprehensive. Various methods for studying their properties with/without the external forcing term on feedback control [26–29], strange attractor [30–33], stability [34–36], periodic solutions [37–40] and numerical simulations [41–43] have been proposed and some profound results have been established. A phase analysis to Eqs. 2 and 3 is presented in [44] and more qualitative studies have been described in [22]. Exact solutions were discussed by Chen using the target function method [45], but explicit solutions were not shown. Note that Eq. 2 satisfies the Painlevé condition with a certain parametric choice [25,46]. In [47], exact solutions to Eq. 3 were presented by using the elliptic function method for various special cases. The harmonic solutions were investigated by McCartin using the method of van der Pol [34]. The behavior of the solutions of the Duffing's equation near the separatrix were treated by Hale and Spezamiglio [48].

On the hand, describing traveling waves of non-linear evolution equations has been one of basic problems in theoretical and experimental physics, and traveling wave solutions to many one-dimensional non-linear evolution equations can be derived from a set of ordinary differential equations that can be interpreted as a flow in a three-dimensional phase space. In the past decade, experiments on one-dimensional states of non-linear traveling wave convection were undertook in a narrow annular cell [49,50]. Spatially uniform states are found to be stable within a band of wave numbers whose width grows approximately as the square root of the distance above a saddle-node Rayleigh number. Inside the band, the static properties of the traveling waves were measured, including their response to spatial inhomogeneities in the Rayleigh number. Outside the stability band, traveling wave states become unstable to temporally growing modulations of the spatial wave-number profile that propagate through the system as the group velocity of the underlying traveling waves. What are the prospects for a theoretical explanation for the results presented in the experiments? The complex Ginzburg–Landau equation (CGLE) is the usual model invoked in theoretical discussions of pattern-forming systems. The lowest-order CGLE which describes a system exhibiting a subcritical bifurcation to traveling waves must contain a quintic non-linearity; at this order, it is necessary to include the lowest-order non-linear gradient terms [51,52]

$$\begin{aligned} u_t = \alpha u + (b_1 + ic_1)u_{xx} - (b_2 - ic_2)|u|^2 u - (b_3 - ic_3)|u|^4 u \\ +(b_4 + ic_4)(|u|^2 u)_x + (b_5 + ic_5)(|u|^2)_x u, \end{aligned} \tag{5}$$

where $u(x, t)$ is a complex function, and b_i, c_i $(i = 1, 2, \ldots, 5)$ and α are real coefficients. This equation can be examined for stability against phase modulations using rather straightforward techniques [49]. However, there is an immediate problem. This equation cannot even account for the non-linear dispersion of the stable traveling wave studied in these experiments. More results on the special subclass of Eq. 5 can be seen in [50–53] and the analysis for traveling wave solutions (or uniformly translating solutions) appears to be dominated by numerical methods, perturbation theory and so on. It does not seem that the detailed analysis of traveling wave solutions of CGLE (5) and their explicitly functional forms have been presented previously.

Since Eq. 1 is not solvable in the general case and does not pass Painlevé test, to analyze the exact solution, qualitative study together with ingenious mathematical techniques appears to be more important. Recently, qualitative results for physical, chemical and biological systems have been studied extensively [54–56]. Our primary goal of this paper is to provide a qualitative analysis to Eq. 1 and find the Lie symmetry under certain parametric choices. In the subsequent work, we will apply the results obtained in this paper to the study of CGLE for its uniformly translating solutions. Several classes of uniformly translating solutions will be established under various parameter conditions.

The organization of this paper is as follows. In Sect. 2, making use of the qualitative theory of dynamical systems, we demonstrate a qualitative analysis to a two-dimensional plane autonomous system which is equivalent to the Duffing–van der Pol non-linear oscillator Eq. 1. In Sect. 3, we show that the Duffing–van der Pol non-linear oscillator admits infinite dimensional symmetry algebras under given parametric choices for which the system becomes integrable. The associated first integrals are given by means of the Lie symmetry method. Sect. 4 is a brief conclusion.

2 Qualitative study

In this section, we give a qualitative analysis to Eq. 1 which indicates that under certain parametric conditions, Eq. 1 has a bounded non-trivial solution. In next section, we will apply the Lie symmetry method to seek the first integral for Eq. 1. By using the first integral, Eq. 1 can be reduced to a first-order ordinary differential equation (ODE), thus the bounded solutions can be found by solving the resultant first-order ODE.

Letting $\frac{dx}{dt} = y$, Eq. 1 is equivalent to the following two-dimensional autonomous system

$$\begin{cases} \dot{x} = P(x,\ y) = y \\ \dot{y} = Q(x,\ y) = -(\alpha + \beta x^2)y + \gamma x - x^3 + \mu x^5. \end{cases} \tag{6}$$

In this section, using the qualitative theory of differential equations, we will show a qualitative result to the Duffing–van der Pol non-linear Eq. 1. Specifically, we show that under certain conditions

$$\alpha < 0, \quad \beta < 0, \quad \mu < 0, \quad \gamma > 0, \tag{7}$$

the bounded exact solution $x(t)$ of Eq. 1 is strictly monotone decreasing with respect to the time t. To prove this, we need to show that $x'(t) \neq 0$ for any $t \in \mathbb{R}$ and $x(-\infty) > x(+\infty)$ under condition (7). That is, the associated orbit in the Poincaré phase plane does not intersect the x-axis for any t.

Consider system (6) in the Poincaré phase plane. Under condition (7), system (6) has three equilibrium points

$$E\left(-\sqrt{\frac{1-\sqrt{1-4\mu\gamma}}{2\mu}},\ 0\right), \quad O(0,\ 0), \quad Q\left(\sqrt{\frac{1-\sqrt{1-4\mu\gamma}}{2\mu}},\ 0\right). \tag{8}$$

Denote $\Delta_1 = \alpha + \beta x_1^2$, $\Delta_2 = \frac{1-4\mu\gamma-\sqrt{1-4\mu\gamma}}{\mu}$, $\Delta_3 = \Delta_1^2 + 4\Delta_2$, where $x_1^2 = \frac{1-\sqrt{1-4\mu\gamma}}{2\mu}$. Analyzing the eigenvalues of the linearization of system (6), we have

(1) when $\Delta_3 \geq 0$, O is a saddle point, E and Q are unstable nodal points;
(2) when $\Delta_3 < 0$, O is a saddle point, E and Q are unstable spiral points.

Moreover, using the Poincaré transformation, we can see that there are two infinite equilibrium points on the y-axis. Since $\frac{\partial P(x,\ y)}{\partial x} + \frac{\partial Q(x,\ y)}{\partial y} = \alpha + \beta x^2 < 0$, according to the Bendixson Theorem ([44]), system (6) has no closed orbit in the Poincaré phase plane. That is, Eq. 1 has neither bell-profile wave solutions, nor periodic solutions. The local behavior at the equilibrium points of system (6) is depicted in Fig. 1.

Notice that in the left scheme of Fig. 1, except the equilibrium points E, Q and the orbits $L(E,\ O)$ and $L(Q,\ O)$, all other orbits in the Poincaré phase plane either emanate from the infinite equilibrium point or approach the infinite equilibrium point as $t \to +\infty$. This implies that the y-coordinates of the points which lies on the orbits except E, O, Q, $L(E,\ O)$ and $L(Q,\ O)$ are unbounded, so are the corresponding x-coordinates on the same orbits. This can be seen by way of contradiction. Assume that there exists a positive number δ such that $|x| < \delta$ as $y \to \infty$. By the Mean-value Theorem, $\frac{dy}{dx}$ is unbounded.

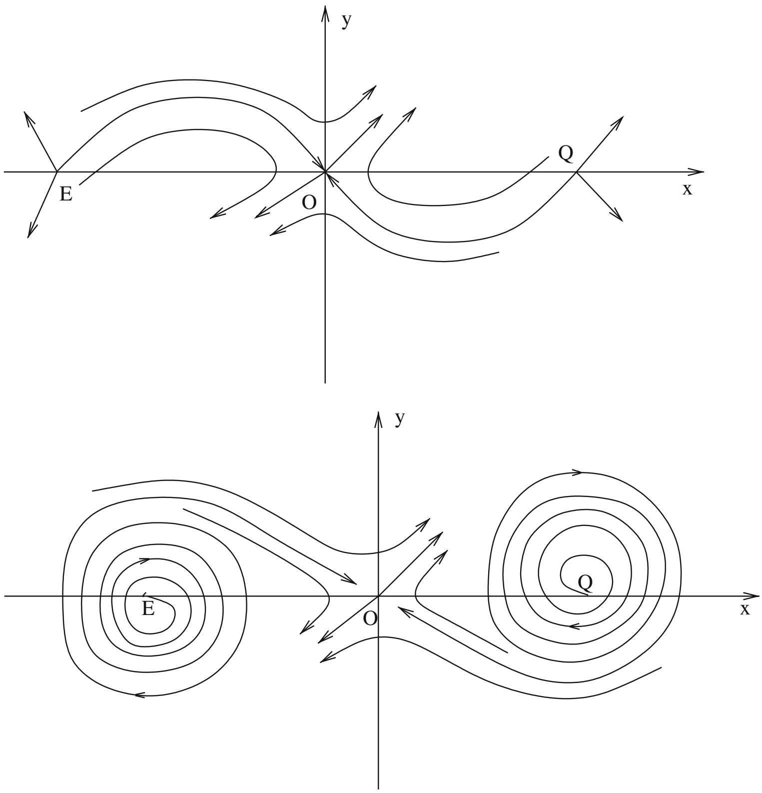

Fig. 1 The local behavior of system (6) in the cases of $\Delta_3 \geq 0$ and $\Delta_3 < 0$, respectively

On the other hand, since the slope of the tangent line to each orbit at the point (x, y) can be expressed

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -(\alpha + \beta x^2) + \frac{\gamma x - x^3 + \mu x^5}{y}. \tag{9}$$

Equation 9 implies that $\frac{\mathrm{d}y}{\mathrm{d}x} < |\alpha + \beta\delta^2| + \frac{1}{2}$ as $y \to \infty$. This yields a contradiction.

Letting $Q(x, y) = 0$, we have

$$y = \frac{\gamma x - x^3 + \mu x^5}{\alpha + \beta x^2}, \tag{10}$$

which is the trajectory on which each orbit points to the left or right. Observe that under conditions (7), expression (10) can be rewritten as

$$y = \frac{\mu x(x^2 - x_1^2)(x^2 + x_2^2)}{\alpha + \beta x^2},$$

where $x_1^2 = \frac{1-\sqrt{1-4\mu\gamma}}{2\mu}$ and $x_2^2 = -\frac{1+\sqrt{1-4\mu\gamma}}{2\mu}$. Note that the graph of Eq. 10 is symmetric about the origin and the derivative of (10) is

$$y' = \frac{3\mu\beta x^6 + 5\alpha\mu x^4 - \beta x^4 - 3\alpha x^2 - \gamma\beta x^2 + \alpha\gamma}{(\alpha + \beta x^2)^2}.$$

This indicates that the curve of Eq. 10 has at most two critical points in quadrant IV.

Construct two lines l_1 and l_2. l_1 is the tangent line of the curve of Eq. 10 at the origin, i.e., $y = \frac{\gamma}{\alpha}x$, and l_2 passes through $Q(x_1, 0)$ with the slope $K = \frac{-\alpha+\sqrt{\alpha^2+8\mu x_1^2(x_1^2+x_2^2)}}{4}$, i.e.,

$$l_2: \quad y = \frac{-\alpha + \sqrt{\alpha^2 + 8\mu x_1^2(x_1^2 + x_2^2)}}{4}(x - x_1). \tag{11}$$

Denote the intersection point of l_1 and l_2 by T, and the x-coordinate of T by x_T. Immediately we have the following

$$0 < x_T = \frac{-\alpha + \sqrt{\alpha^2 + 8\mu x_1^2(x_1^2 + x_2^2)}}{4} \frac{x_1}{K - (\gamma/\alpha)} < x_1. \tag{12}$$

Note that the curve of Eq. 10 is above the tangent line l_1 in quadrant IV. Denote by Ω the convex domain bounded by line segments OQ, QT and TO (see Fig. 2). It suffices to prove that any integral orbit which starts from a point outside the convex domain Ω, cannot re-enter the convex domain Ω as $t \to \pm\infty$. It is obvious that there is no integral orbit entering Ω through OQ, because all orbits at each point of the x-axis between O and Q are orthogonally along the direction of the positive y-axis. In addition, at the each point of the line segment OT, we have

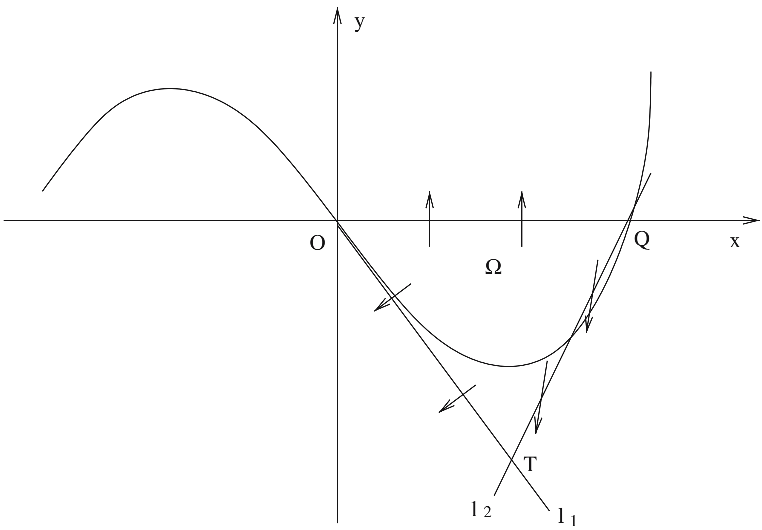

Fig. 2 The convex domain Ω bounded by OQT

$$\frac{dy}{dx}\Big|_{(x,\, y)\in OT} = -\alpha - \beta x^2 + \frac{\gamma x - x^3 + \mu x^5}{(\gamma/\alpha)x}\Bigg|_{(x,\, y)\in OT}$$
$$= -\beta x^2 - \frac{\alpha}{\gamma}x^2 + \frac{\alpha\mu}{\gamma}x^4$$
$$> \frac{\gamma}{\alpha}.$$

This implies that all orbits at each point on the line segment OT point outward. Similarly, on the line segment QT, using (11), we have

$$\frac{dy}{dx}\Big|_{(x,\, y)\in l_2} = -(\alpha + \beta x^2) + \frac{\mu x^5 - x^3 + \gamma x}{y}\Bigg|_{(x,\, y)\in QT}$$
$$= -(\alpha + \beta x^2) + \frac{\mu x(x^2 - x_1^2)(x^2 + x_2)}{K(x - x_1)}\Bigg|_{(x,\, y)\in QT}$$
$$> -\alpha - \beta X_T^2 + \frac{\mu x(x + x_1)(x^2 + x_2^2)}{K}.$$

Since $x_T \leq x \leq x_1$, from (12), we obtain

$$\frac{dy}{dx}\Big|_{(x,\, y)\in QT} > -\alpha - \beta x_T^2 + \frac{2\mu x_1^2(x_1^2 + x_2^2)}{K}$$
$$> \frac{-\alpha + \sqrt{\alpha^2 + 8\mu x_1^2(x_1^2 + x_2^2)}}{4} = K.$$

This implies that except at Q, all orbits at each point of line segment QT point outward. Therefore, the orbit $(Q,\ O)$ must lie inside the convex domain Ω. In other words, except the end points O and Q, the orbit $L(Q,\ O)$ cannot intersect the x-axis. Hence, $x'(t) \neq 0$. Furthermore, from (8), we have

$$\lim_{t\to+\infty} x(t) = x(+\infty) = 0, \qquad \lim_{t\to-\infty} x(t) = x(-\infty) = \sqrt{\frac{1 - \sqrt{1 - 4\mu\gamma}}{2\mu}}.$$

Therefore, we conclude that under condition (7) and $\Delta_3 \geq 0$, the bounded exact solution $x(t)$ associated with the orbit $L(Q,\ O)$ is strictly monotone decreasing with respect to t.

3 Lie symmetries

Painlevé and his colleagues [46] considered the problem of classifying differential equations whose solutions, as functions of a complex variable, have only poles as movable (i.e., dependent upon initial conditions) singularities. It has been shown that equations with this Painlevé property are more likely to be explicitly solvable. For a second-order ordinary differential equation, it has been known that under what conditions such an equation could be reduced to a first-order equation whose solutions were free of movable critical points. However, as we know, many second-order ordinary differential equations do not pass the Painlevé test, but they possess one first integral and hence are integrable.

During the last century, the integrability problem of non-linear differential equations and fascinating qualitative phenomena have attracted the attention of both experimentalists and theorists. One of the most innovative and powerful methods for analyzing and computing non-linear partial differential equations is the Lie symmetry method [57–59]. As clearly described in [57], a significant feature of this approach is that in many non-linear problems, one can derive special solutions associated with non-linear differential equations straightforwardly which are otherwise inaccessible through other existing methods [60–62]. The basic idea of the Lie symmetry method is to seek the corresponding symmetry groups associated with a given differential equation under a continuous group of transformations and to find a reduction transformation from the symmetries. For the case of partial differential equations, the reduction transformation can be used to reduce the number of independent variables by one; for example, a PDE with two independent variables to an ordinary differential equation (ODE). For a reduced ODE, one can check whether it is of Painlevé type or not and it is often the case that when the reduced ODE is of Painlevé type it can be solved explicitly thereby leading to a solution of the original PDE [63]. In this section we show that the first integral of system (6) can be obtained by means of the Lie symmetry method with certain parameter choices.

In order to find Lie symmetry, we need to look for the invariance of (6) under a one-parameter infinitesimal point transformations of the form

$$\begin{aligned} X_i &= x_i + \epsilon\eta_i(t,\, x_i), \qquad i = 1,\, 2 \\ T &= t + \epsilon\xi(t,\, x_i). \end{aligned} \tag{13}$$

The corresponding infinitesimal generator is

$$V = \xi(t,\, x_i)\frac{\partial}{\partial t} + \eta(t,\, x_i)\frac{\partial}{\partial t}. \tag{14}$$

In order to find the first prolongation of the vector V, we take $\xi = 0$ in (14) and define the corresponding first extended operator

$$Pr^{(1)}V = \eta_i\frac{\partial}{\partial x_i} + \dot{\eta}_i\frac{\partial}{\partial \dot{x}_i},$$

where $\dot{\eta}_i = D_t\eta_i,\ i = 1,\ 2$ and D_t is the total differential operator. V is called the generator of a one-parameter symmetry group for system (6) if, whenever system (6) is satisfied and

$$Pr^{(1)}V(\triangle_i)|_{(3)} = \left(\eta_i\frac{\partial}{\partial x_i} + \dot{\eta}_i\frac{\partial}{\partial \dot{x}_i}\right)(\triangle_i) = 0,$$

where the $\triangle_i' s$ $(i = 1,\, 2)$ denote two equations in (6).

The invariance requirement of (6) under the infinitesimal transformations (13) can be expressed as

$$\begin{cases} \dot{\eta}_1 = \eta_2 \\ \dot{\eta}_2 = \eta_1(-2\beta xy + \gamma - 3x^2 + 5\mu x^4) - \eta_2(\alpha + \beta x^2). \end{cases} \tag{15}$$

In order to identify a non-trivial infinite dimensional Lie algebra of symmetry vector fields which can be directly associated with the integral of motion for a suitable parametric choice, we may start our study by assuming that the Lie symmetries η_i $(i = 1,\, 2)$ in (15) have the quadratic form

$$\eta_1 = a_1 + a_2 y + a_3 y^2,$$
$$\eta_2 = b_1 + b_2 y + b_3 y^2, \tag{16}$$

where the $a_i's$ and $b_i's$ ($i = 1, 2, 3$) are functions of t and x. Substituting (16) into (15) and equating the coefficients of various powers of y, one can get the resultant determining equations

$$a_{3x} = 0, \tag{17}$$
$$b_{3x} + 2\beta a_3 x = 0, \tag{18}$$
$$a_{3t} + a_{2x} - 2(\alpha + \beta x^2)a_3 - b_3 = 0, \tag{19}$$
$$b_{3t} + b_{2x} - (\alpha + \beta x^2)b_3 - (\gamma - 3x^2 + 5\mu x^4)a_3 + 2\beta a_2 x = 0,$$
$$a_{2t} + a_{1x} - (\alpha + \beta x^2)a_2 + 2(\gamma x - x^3 + 5\mu x^4)a_3 - b_2 = 0, \tag{20}$$
$$b_{2t} + b_{1x} + 2(\gamma x - x^3)b_3 - (\gamma - 3x^2 + 5\mu x^4)a_2 + 2\beta a_1 x = 0,$$
$$a_{1t} + (\gamma x - x^3 + 5\mu x^4)a_2 - b_1 = 0, \tag{21}$$
$$b_{1t} - (\gamma x - x^3 + 5\mu x^4)b_2 - (\gamma - 3x^2 + 5\mu x^4)a_1 + (\alpha + \beta x^2)b_1 = 0.$$

Assuming that $a_3 = a(t)$, and from (17)–(21), we have

$$\mu\beta a(t) = 0$$
$$2\alpha - \frac{6}{\beta} + \frac{2}{3}\gamma\beta = 0$$
$$\frac{6}{\beta}(\alpha^2 - 2\gamma) - \frac{54\alpha}{\beta^2} + \frac{108}{\beta^3} + 2\alpha\gamma = 0.$$

Case 1 When $\mu = 0$ and $\beta \neq 0$, under the parametric choices

$$\alpha \cdot \beta = 4, \quad \beta^2 \cdot \gamma = -3, \tag{22}$$

with the aid of Maple, we can solve the determining equations consistently and obtain non-trivial forms for the functions $a_i's$ and $b_i's$. Using these functions, we obtain a four-parameter symmetry group for system (6), and then the corresponding four vector fields are given as follows:

$$S_1 = X = y\frac{\partial}{\partial x} - \left[(\beta x^2 + \frac{4}{\beta})y + x^3 + \frac{3}{\beta^2}x\right]\frac{\partial}{\partial y},$$
$$S_2 = \mathrm{e}^{(2/\beta)t}\left((y + \frac{x}{\beta})\frac{\partial}{\partial x} - \left[(\beta x^2 + \frac{1}{\beta})y + x^3 + \frac{x}{\beta^2}\right]\frac{\partial}{\partial y}\right),$$
$$S_3 = \mathrm{e}^{6t/\beta}\left(y^2 + y\left[\frac{2}{3}\beta x^3 + \frac{2x}{\beta} + c_1\mathrm{e}^{-3t/\beta}\right]\right.$$
$$\left. + x^2\left[\frac{1}{9}\beta^2 x^4 + \frac{2}{3}x^2 + \frac{1}{\beta^2}\right] + c_1 x\mathrm{e}^{-3t/\beta}\left[\frac{1}{3}\beta x^2 + \frac{1}{\beta}\right]\right) S_1,$$
$$S_4 = \mathrm{e}^{6t/\beta}\left(y^2 + y\left[\frac{2}{3}\beta x^3 + \frac{2x}{\beta} + c_1\mathrm{e}^{-3t/\beta}\right]\right.$$
$$\left. + x^2\left[\frac{1}{9}\beta^2 x^4 + \frac{2}{3}x^2 + \frac{1}{\beta^2}\right] + c_1 x\mathrm{e}^{-3t/\beta}\left[\frac{1}{3}\beta x^2 + \frac{1}{\beta}\right]\right) S_2,$$

where X is the dynamical vector field. Since the vector fields S_3 and S_4 are not functionally independent, one can use them to generate the integral of motion associated with the dynamical system (6) as

$$e^{6t/\beta}\left(y^2 + y\left[\frac{2}{3}\beta x^3 + \frac{2x}{\beta} + c_1 e^{-3t/\beta}\right] + x^2\left[\frac{1}{9}\beta^2 x^4 + \frac{2}{3}x^2 + \frac{1}{\beta^2}\right] + c_1 x e^{-3t/\beta}\left[\frac{1}{3}\beta x^2 + \frac{1}{\beta}\right]\right) = c_2. \quad (23)$$

Letting $c_1 = 2c$ and $c_2 = -c^2$, we obtain a particular case of (23) as

$$e^{(3/\beta)t}\left(y + \frac{\beta}{3}x^3 + \frac{x}{\beta}\right) + c = 0. \quad (24)$$

Combining (24) and (6), under condition (22) we can reduce Eq. 1 to an Able equation of the first kind

$$\dot{x} + \frac{1}{3}\beta x^3 + \frac{1}{\beta}x = -ce^{-\frac{3}{\beta}t}. \quad (25)$$

Case 2 When $\alpha = 0$ and $\beta = 0$, apparently we can obtain one first integral to Eq. 1 for arbitrary γ and μ as

$$\dot{x} - \sqrt{\gamma x^2 - \frac{x^4}{2} + \frac{\mu x^6}{3} + C_0} = 0, \quad (26)$$

where C_0 is arbitrary real constant.

Equations 25 and 26 are very useful when we solve CGLE for its uniformly translating solutions. They provide one of mathematical bases of further work [64], in which several uniformly translating solutions under various parameter conditions are expressed in terms of hyperbolic functions, implicit functions and trigonometrical functions, respectively.

4 Conclusion

One of the most fundamental equations in the study of non-linear oscillations is the Duffing–van der Pol equation, which has been discussed in many works, for different systems arising in various scientific fields. Due to the occurrence of two higher-order non-linear terms, many basic problems on the Duffing–van der Pol equation have not been solved yet in the general case. Therefore, qualitative analysis as well as powerful mathematical techniques seems to be more important.

In this paper, we apply the qualitative theory of planar systems to study a two-dimensional plane autonomous system which is equivalent to the Duffing–van der Pol Eq. 1. The monotone property of the bounded exact solution is found. One first integral to the Duffing–van der Pol equation is obtained by using the Lie symmetry method. Some applications of these qualitative results to the study of uniformly translating solutions of the complex Ginzburg–Landau equation will be demonstrated in a subsequent work.

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